A Pedagogical Approach to General Relativity and the Membrane Paradigm

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Abstract
General Relativity is one of the most important theories of modern physics; completely changing how we understand the universe. We have prepared a comprehensive introduction to the mathematical and graphical tools used in calculations using the theory of General Relativity, as well as present a specific example of dropping a small charge into a black hole in order to demonstrate some of the nuances of General Relativity that give rise to the Information Paradox.

Describing Space-time
Distances in Space-time can be described using the Line Element. For all spacetime geometries, the line element can be calculated with the following:

\[ dt^2 = g_{\mu\nu} dx^\mu dx^\nu \]

Where \( g_{\mu\nu} \) is the metric tensor describing the particular spacetime geometry being studied. For flat, empty space, the metric tensor is very simple.

\[ g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

For other spacetime geometries, such as the Schwarzschild Geometry, which describes the spacetime curvature around a black hole, will have more complicated metrics.

The Schwarzschild Geometry
The spacetime geometry around a black hole can be described with the following metric tensor.

\[ g_{\mu\nu} = \begin{pmatrix} (1 - 2M/r) & 0 & 0 & 0 \\ 0 & (1 - 2M/r)^{-1} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \]

From this second term in our metric, we can easily see that we have singularities at \( r = 0 \) and \( r = 2M \). This point at \( r = 2M \) is called the "Event Horizon" and has some interesting properties which we can see by applying some coordinate transformations.

World Lines
The path a particle traces through spacetime is called its "World Line". In terms of the line element, there are three possibilities for any such world line:

1. \( dr^2 < 0 \): Time-like Separation
2. \( dr^2 = 0 \): Null Separation
3. \( dr^2 > 0 \): Space-like Separation

In order to understand some spacetime geometries, it can be useful to look at the paths that light rays follow in these different geometries. Any two events who have null separation are such that you can only reach the other moving at light speed. So, solving for the equations of motion when \( dr^2 = 0 \) gives us the paths that light move through space in these geometries.

The Stretched Horizon
From the point of a FIDOS all things falling into the horizon will never cross it, but instead, after a very long time, approach the surface a very small distance above it, called the "Stretched Horizon".

After deriving Maxwell's equations in General Relativity, we can then represent the surface charge on the horizon as...

\[ \sigma = \frac{1}{4\pi} E \]

Taking a time derivative and applying Maxwell's equations, we can make the substitution \( \beta = \rho B \) as well as imposing the continuity conditions

\[ 4\pi j = \rho \beta \text{ as well as imposing the continuity conditions} \]

Plugging this relation back into our continuity conditions we obtain

\[ j = \frac{1}{4\pi} E \]

Which is a variation of Ohm's Law, \( j = \sigma E \), where \( \sigma \) is the conductivity! Thus, we have shown that the stretched horizon acts as an ohmic conductor!

Types of Observers
Imagine that you are stationary some distance away from a black hole and your friend, who began by your side, begins to fall in towards the black hole. By switching to Rindler coordinates, characterized by \( r \rightarrow p \), which describes the proper distance to the horizon, and \( t \rightarrow 4MGu \), we can simplify our Schwarzschild line element to

\[ dr^2 = p^2 d\omega^2 - dp^2 - dx^2 - dy^2 \]

By the relative sign difference between \( d\omega \) and \( dp \) tells us that "p and \( \omega \) are radial hyperbolic angle variables of an ordinary Minkowski space" [Susskind].

Applying a boost in the \( \rho \) direction, as well as rescaling our Rindler time coordinate we can transform our viewpoint to our free-falling observer (PREFOS). Doing this, we return to a form identical to Minkowski space!

\[ T = \rho \sinh(\omega) \; ; \; Z = \rho \cosh(\omega) \]

This tells us that a free-falling observer will not see the spacetime geometry of the black hole, but rather, empty, flat spacetime. So, what happened to the event horizon we see in the FIDOS point of view? Is the event horizon a real thing or simply a construct of our choice of coordinates?

The Physical Horizon
Applying yet another coordinate transformation we can resolve the singularity at \( r = 2M \).

\[ t = \rho - r - 2M \log \left( \frac{r}{2M} - 1 \right) \]

It is important to notice that this is not a different geometry, but simply a change in coordinates, so the same physical properties still hold. By looking at points where \( dr^2 = 0 \) we can see the paths that light will follow; these are points have, what we call, null separation.

Plotting these light cones out, with \( \rho \) on the horizontal axis and \( \omega \) (rindler time) on the vertical axis we can see that the effects of the event horizon are still present!

The Membrane Paradigm
Using this definition of the surface charge density, if we see it equal to \( \frac{1}{4\pi} \) (total change over surface area) and solve for \( \omega \), we can calculate the time it takes for the charge to spread over the entire horizon. Doing this we obtain

\[ \omega = \log(2M - \rho_0) \]

Most objects thermalize on timescales proportional to a power of the entropy. This thermalization time is proportional to the log of the entropy, making it the fastest possible in the universe.

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