A Pedagogical Approach to General Relativity and the Membrane Paradigm

Symon Kasperek, Arthur Kraus, Joseph Summers
(Under the direction of Dr. Walter Tangarife)

Abstract

General Relativity is one of the most important theories of modern physics; completely changing how we understand the universe. We have prepared a comprehensive introduction to the mathematical and graphical tools used in calculations using the theory of General Relativity, as well as present a specific example of dropping a small charge into a black hole in order to demonstrate some of the nuances of General Relativity that give rise to the Information Paradox.

Describing Space-time

Distances in Space-time can be described using the Line Element. For all spacetime geometries, the line element can be calculated with the following.

$$d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Where $g_{\mu\nu}$ is the metric tensor describing the particular spacetime geometry being studied. For flat, empty space, the metric tensor is very simple.

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For other spacetime geometries, such as the Schwarzschild Geometry, which describes the spacetime curvature around a black hole, will have more complicated metrics.

The Schwarzschild Geometry

The spacetime geometry around a black hole can be described with the following metric tensor.

$$g_{\mu\nu} = \begin{bmatrix} -(1-2M/r) & 0 & 0 & 0 \\ 0 & (1-2M/r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix}$$

From this second term in our metric, we can easily see that we have singularities at r = 0 and r = 2M. This point at r = 2M is called the "Event Horizon" and has some interesting properties which we can see by applying some coordinate transformations.

World Lines

The path a particle traces through spacetime is called its "World Line". In terms of the line element, there are three possibilities for any such world line:

1. $d\tau^2 < 0$: Time-like Separation
2. $d\tau^2 = 0$: Null Separation
3. $d\tau^2 > 0$: Space-like Separation

In order to understand some spacetime geometries, it can be useful to look at the paths that light rays follow in these different geometries. Any two events who have null separation are such that you can only reach the other moving at light speed. So, solving for the equations of motion when $d\tau^2 = 0$ gives us the paths that light move through space in these geometries

Types of Observers

Imagine that you are stationary some distance away from a black hole and your friend, who began by your side, begins to fall in towards the black hole. By switching to Rindler coordinates, characterized by $r \longrightarrow \rho$, which describes the proper distance to the horizon, and $t \longrightarrow 4MG\omega$, we can simplify our Schwarzschild line element to

$$d\tau^2 = \rho^2 d\omega^2 - d\rho^2 - dx^2 - dy^2$$

By the relative sign difference between $d\omega$ and $d\rho$ tells us that "ρ and ω are radial hyperbolic angle variables of an ordinary Minkowski space" [Susskind].

Applying a boost in the ρ direction, as well as rescaling our Rindler time coordinate we can transform our viewpoint to our free-falling observer (FREFOS). Doing this, we return to a form identical to Minkowski space!

$$T = \rho \sinh(\omega)\ ,\ Z = \rho \cosh(\omega)$$
$$\Longrightarrow d\tau^2 = dT^2 - dZ^2 - dx^2 - dy^2$$

This tells us that a free-falling observer will not see the spacetime geometry of the black hole, but rather, empty, flat spacetime. So, what happened to the event horizon we see in the FIDOS point of view? Is the event horizon a real thing or simply a construct of our choice of coordinates?

The Physical Horizon

Applying yet another coordinate transformation we can resolve the singularity at r=2M.

$$t = v - r - 2M \log\left|\frac{r}{2M} - 1\right|$$

It is important to notice that this is not a different geometry, but simply a change in coordinates, so the same physical properties still hold. By looking at points where $d\tau^2 = 0$ we can see the paths that light will follow; these are points have, what we call, *null separation*.

Plotting these light cones out, with ρ on the horizontal axis and ω (rindler time) on the vertical axis we can see that the effects of the event horizon are still present!

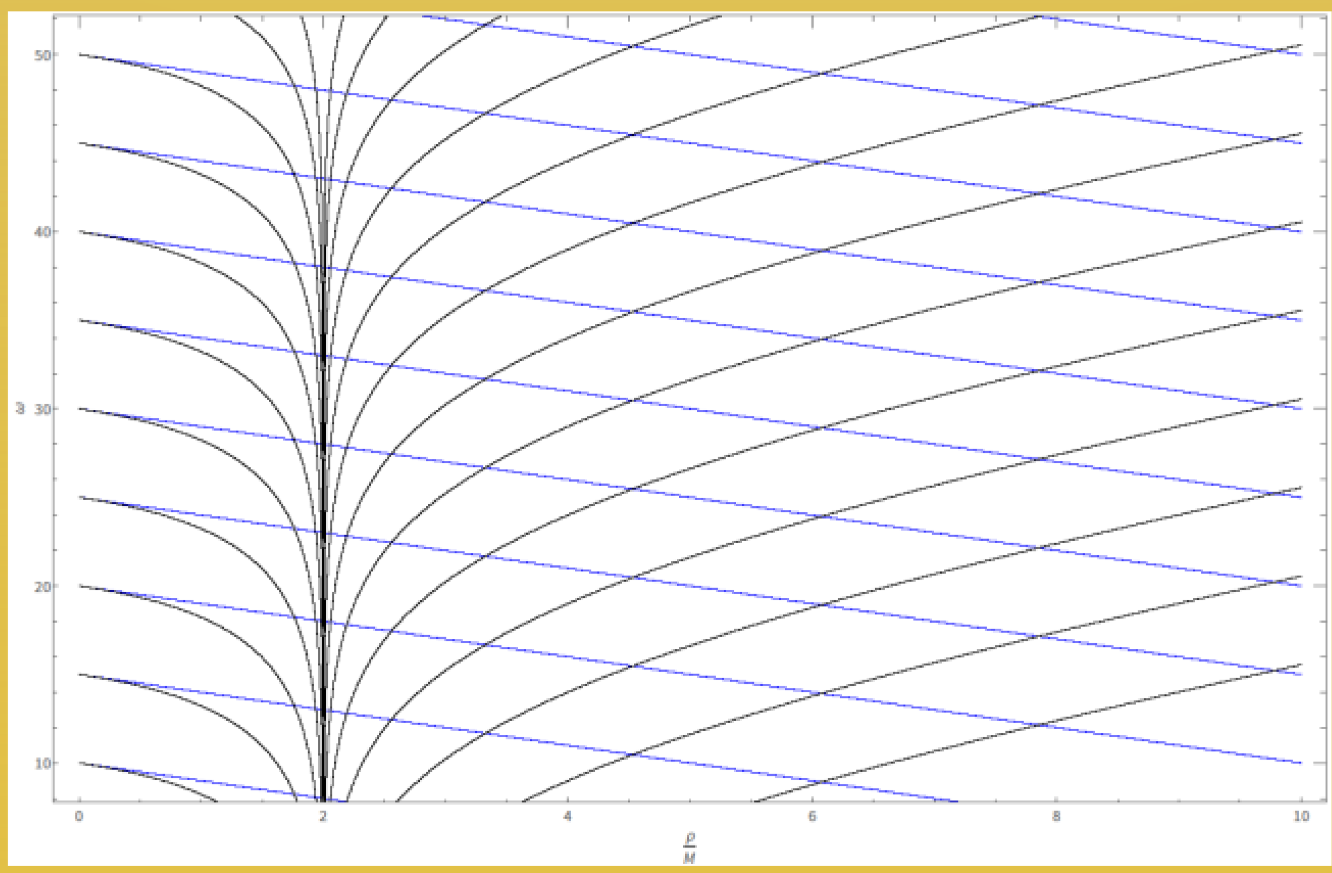

FIG. 1: Light Cones in Schwarzschild Geometry showing that outgoing light rays inside r=2M cannot escape and fall inward.
Black = Radially outward rays
Blue = Radially inward rays

The Stretched Horizon

From the point of a FIDOS all things falling into the horizon will never cross it, but instead, after a very long time, approach the surface a very small distance above it, called the "Stretched Horizon".

After deriving Maxwell's equations in General Relativity, we can then represent the surface charge on the horizon as…

$$\sigma = \frac{1}{4\pi\rho} E_\rho \Big|_{\rho=\rho_0}$$

Taking a time derivative and applying Maxwell's equations, we can make the substitution $\vec{\beta} = \rho\vec{B}$ as well as imposing the continuity conditions

$$4\pi jy = \rho\beta_x \ \&\ 4\pi jx = -\rho\beta_y$$

we can obtain the following relations.

$$\beta_x = E_y \ \&\ \beta_y = -E_x$$

Plugging this relation back into our continuity conditions we obtain

$$j_i = \frac{1}{4\pi} E_i$$

Which is a variation of Ohm's Law, $J = \sigma E$, where σ is the conductivity! Thus, we have shown that the stretched horizon acts as an ohmic conductor!

Dropping a Charge

Now that we understand the how the stretched horizon will react to charge, let us drop some point charge onto it and see what happens! We can right the electric field due to our point charge as

$$\vec{E} = \frac{q}{4\pi\varepsilon_0} \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|^3}$$

Then focusing only on the z-component, and applying the following substituitions to give our charge a boost toward the black hole, we can obtain a relation between E_z and E_ρ.

$$E_z = \frac{(z-z_0)}{[(z-z_0)^2 + x_\perp^2]^{3/2}}\ , T = \rho \sinh(\omega), z = \rho\cosh(\omega)$$
$$\Longrightarrow E_z = \frac{1}{\rho} E_\rho$$

Therefore, our equation for the surface charge density becomes

$$\sigma = \frac{e}{4\pi} \frac{\rho_0 \cosh(\omega) - z_0}{[(\rho_0\cosh(\omega))^2 + x_\perp^2]^{3/2}}$$

Letting ω become really large and rescaling $x_\perp = e^\omega y_\perp$, our surface charge density will simpify to

$$\sigma = \frac{e}{4\pi} \frac{\rho_0 e^{-2\omega}}{[\rho_0^2 + y_\perp^2]^{3/2}}$$

Thermalization Time

Using this definition of the surface charge density, if we se it equal to $\frac{e}{SA}$ (total charge over surface area) and solve for ω, we can calculate the time it takes for the charge to spread over the entire horizon. Doing this we obtain

$$\omega = \log(2MG - \rho_0)$$
$$t = 4MG\log(2MG - \rho_0)$$

Most objects thermalize on timescales proportional to a power of the entropy. This thermalization time is proportional to the log of the entropy, making it the fastest possible in the universe.

Contact

Joseph Summers, Arthur Kraus, Szymon Kasperek
Loyola University Chicago: Physics Department
Email: Jsummers1@luc.edu, akraus3@luc.edu, skasperek@luc.edu

References

1. Kip S. Thorne, Richard H. Price, and Douglas A. Macdonald. Black Holes: The Membrane Paradigm (Yale University Press 1986).
2. A. Einstein, The Foundation of the General Theory of Relativity (Annalen der Physik, 1916).
3. J. B. Hartle, Gravity : An Introduction to Einstein's General Relativity (Addison Wesley, 2003).
4. S. Hawking, Particle Creation by Black Holes (Communications in Mathematical Physics, 1974).
5. L. Barnett, The Universe and Dr.Einstein (The Camelot Press Ltd., 1949).